A Universally Optimal Method for Minimizing Heterogeneously Smooth and Convex Compositions

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Where do we start?

Compositions?

Heterogeneity?

Smoothness...
Convexity...
Generalized?

Universally
Optimal
Guarantees?

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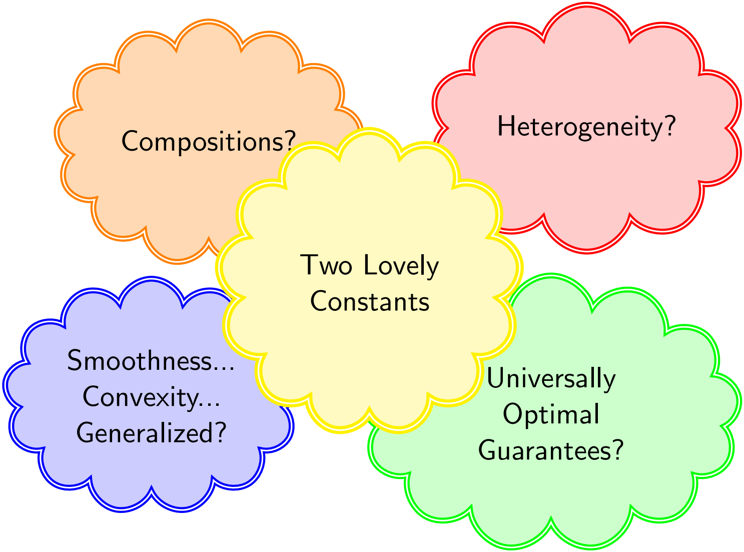
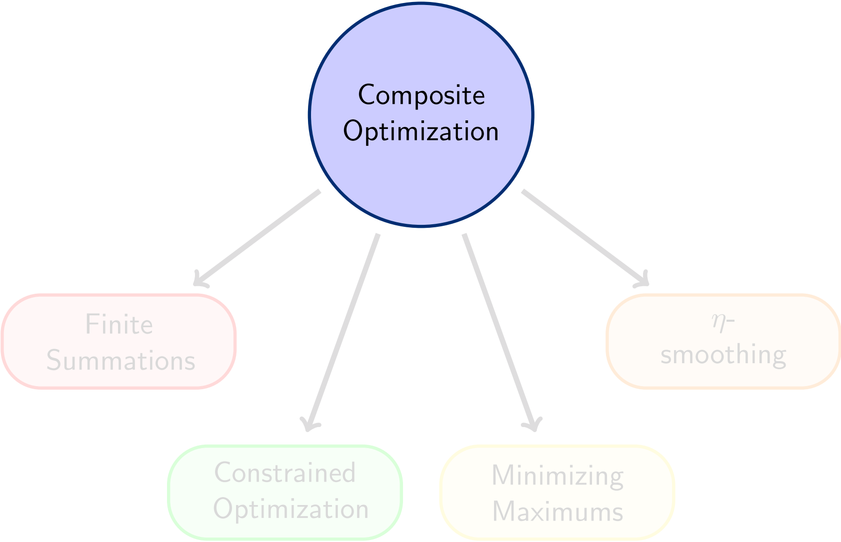


Compositions and Heterogeneity

Composite is an Amazing Model



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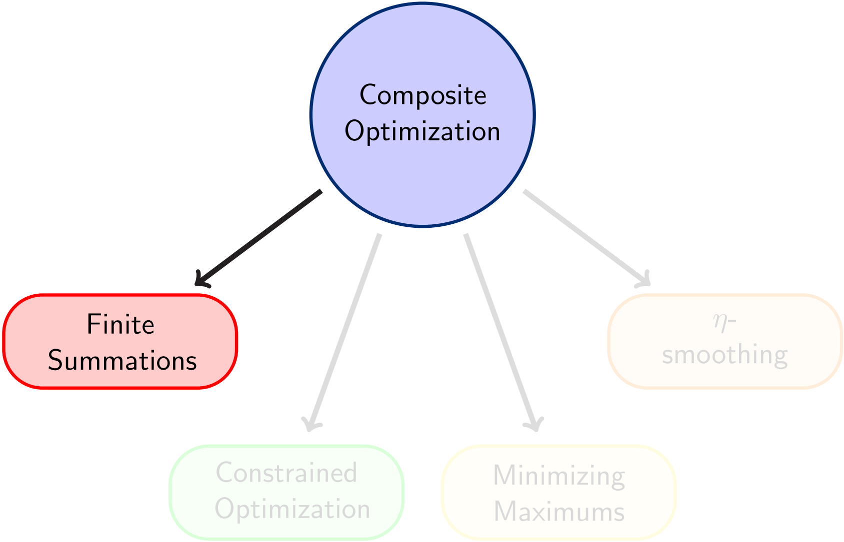

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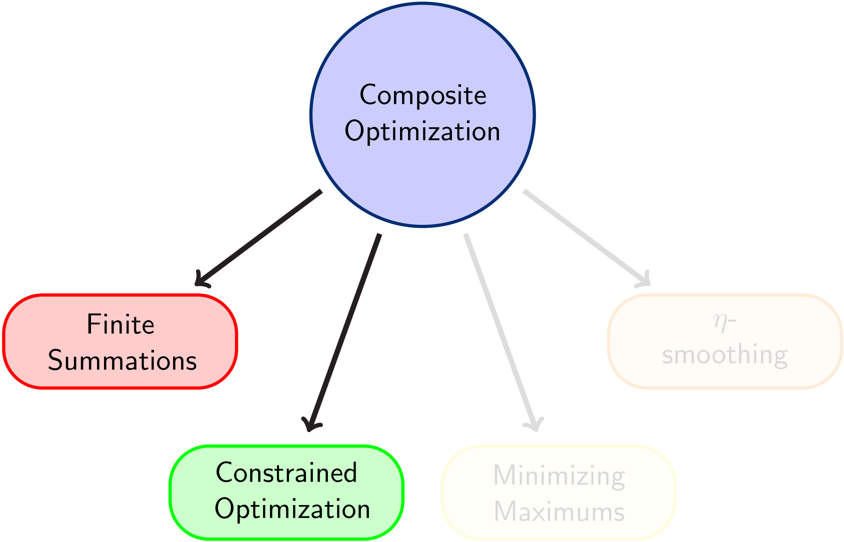

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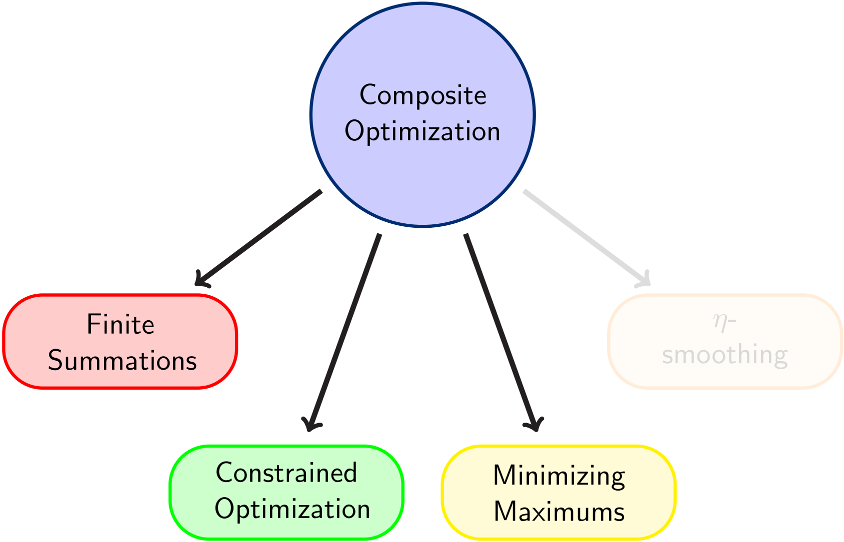

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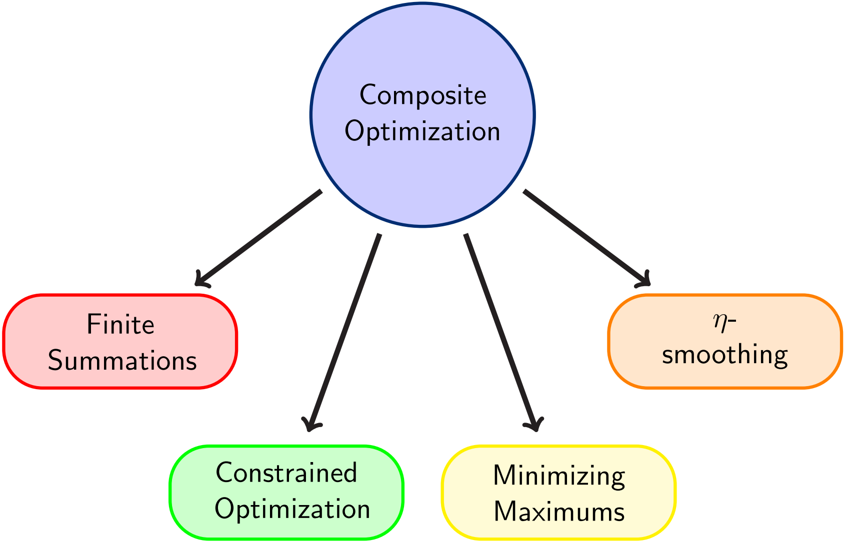

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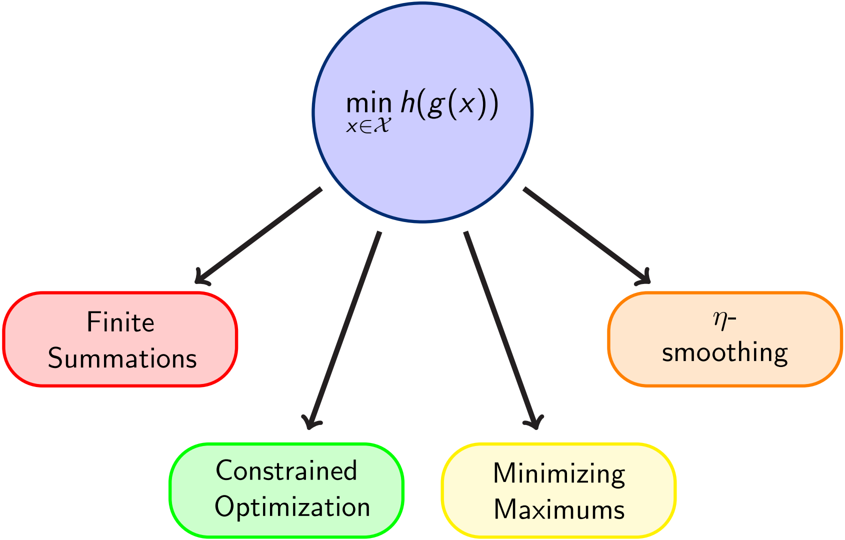

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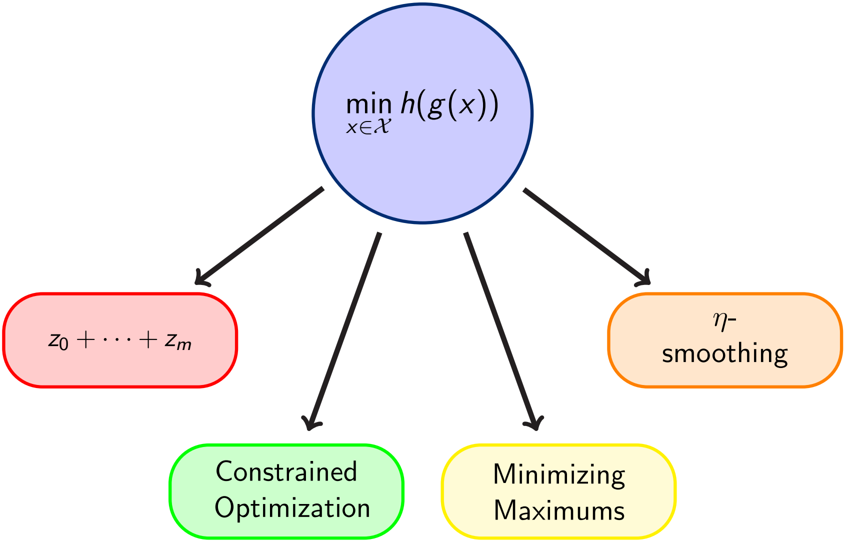

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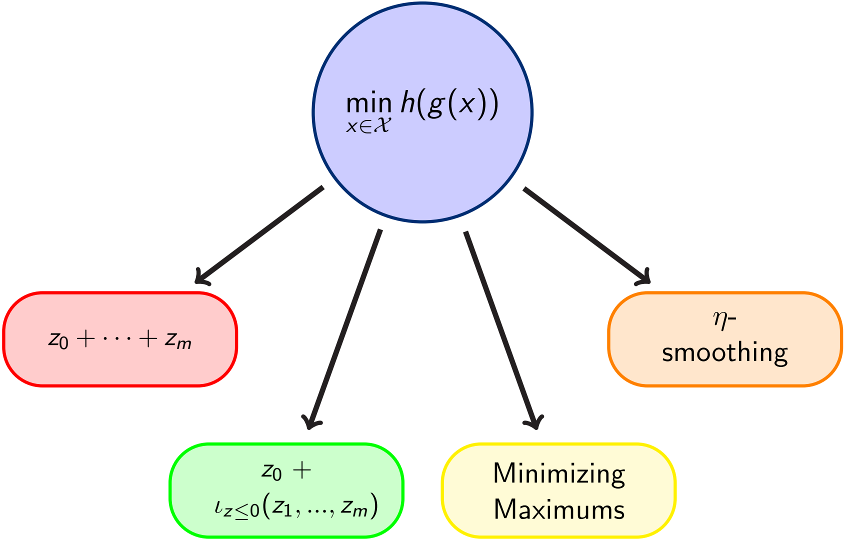

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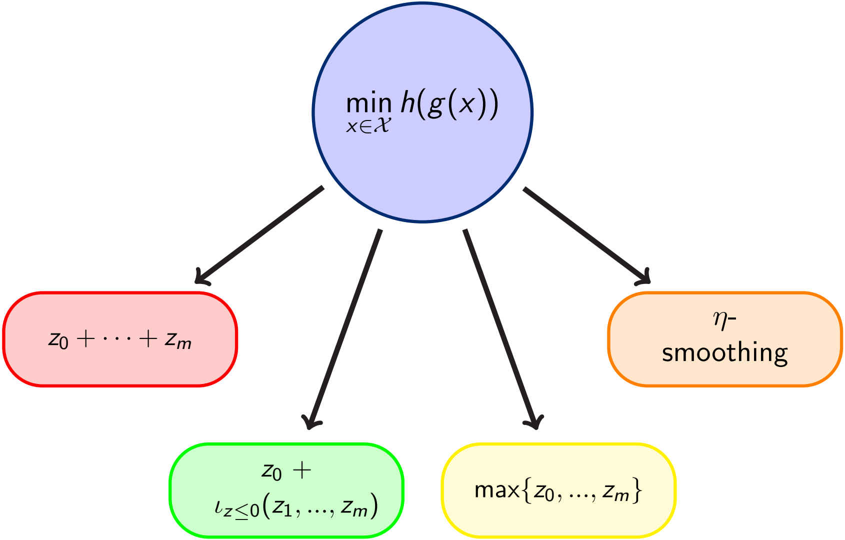

Composite is an Amazing Model

$$\min_{x\in\mathcal{X}} h(g(x))$$

$$z_0 + \cdots + z_m$$

$$z_0 + \iota_{z\le 0}(z_1, \ldots, z_m)$$

$$\max\{z_0, \ldots, z_m\}$$

$$\frac{1}{\eta}\log\sum_{j=0}^{m} e^{\eta z_j}$$

Heterogeneity is an Amazing Model

Consider the following ML problem of **support vector machines**

$$\begin{cases} \min\limits_{w,b,\xi} & \|w\|_2^2 + C\sum_{i=1}^{n} \xi_i \\ s.t. & y_i(w^T x_i - b) \geq 1 - \xi_i \\ & \xi_i \geq 0 \end{cases}$$

$$\min_{w,b} \|w\|_2^2 + C\sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i - b)\}$$

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Heterogeneity is an Amazing Model

We may have some adversarial type problem

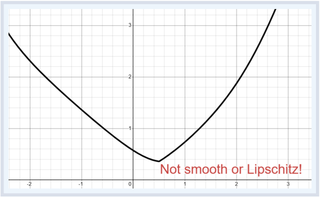
$$\min_{x \in \mathcal{X}} \max_{p \in \{1, \frac{4}{3}, \frac{3}{2}, 2\}} \left\{ \|Ax - b\|_p^p \right\}$$

Heterogeneity is an Amazing Model

Perhaps we're analyzing a mixture model or log-likelihood

$$\min_{x \in \mathcal{X}} \log\left(\sum_{j=1}^{m} \exp(g_j(x))\right)$$

$$\begin{aligned} g_1(x) &= x^2 \\ g_2(x) &= |2x - 1| \end{aligned}$$



Note: Structured components $\implies$ structured objective!

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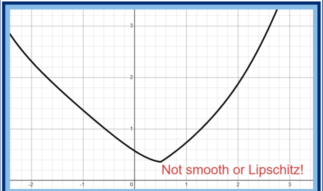

Note: Structured components $\Longrightarrow$ structured objective!

Smoothness and Convexity (Generalized)

Two Dual Notions

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$$

"L-smoothness"

"μ-strong convexity"

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|y - x\|^2$$

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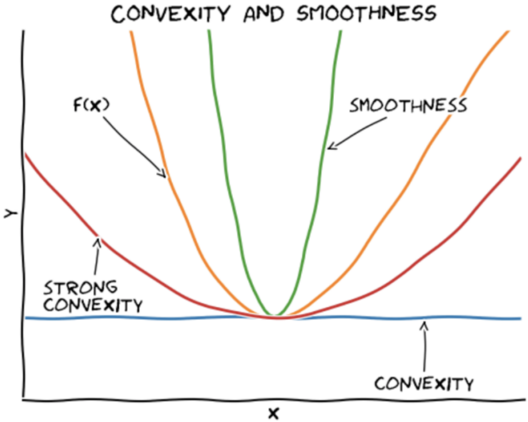
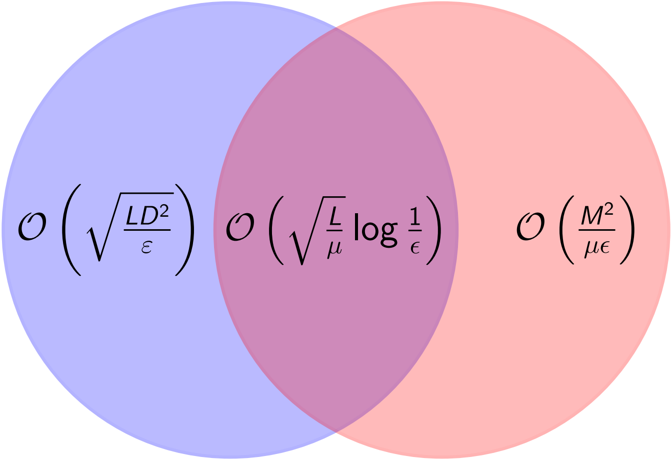


Figure: [2] Smoothness and Strong Convexity Visualized

Smooth and Strong Convexity Rates

L-**smooth** μ-**Strongly convex**[1]



[1]In the nonsmooth case, we assume f is Lipschitz with rank M

Generalizing Smoothness

Smoothness seems pretty restrictive.

Can we generalize, still using first order information?

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(L, p)-Hölder Smoothness

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|^p$$

(L, p)-Hölder continuous gradient

$(p + 1)$-degree upper bound

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{1+p}\|y - x\|^{1+p}$$

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Can we analogously generalize the strong convexity?

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Visual Interlude

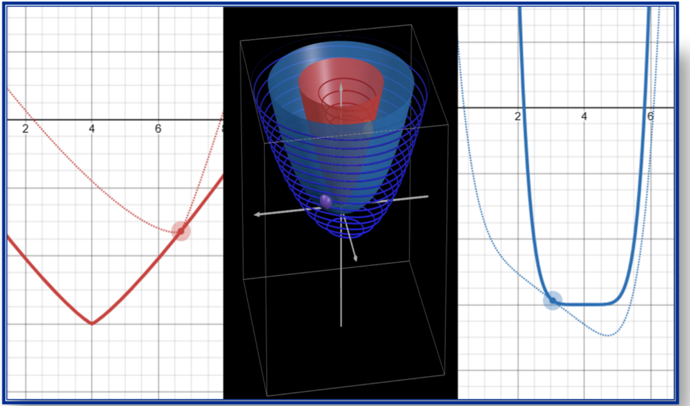

Figure: 2-dim plot and 3-dim plot

Universal Guarantees?

Recall, we minimize $F(x) = h(g_1(x), \ldots, g_m(x))$. Each component having its own (L_j, p_j)-Hölder smoothness and (μ_j, q_j)-uniform convexity.

Suppose we're given magic $\tilde{L}$ and $\tilde{\mu}$ that captures all the information for upper/lower curvature...

What guarantees should we hope for?

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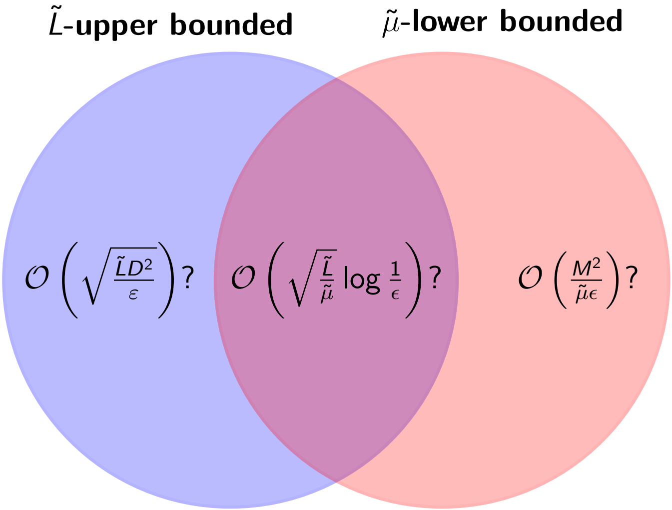
Aggregated Smooth and Strong Convexity Rates?



Motivating our Lovely Constants

From Constraints to Compositions

Constrained Optimization $\longrightarrow$ $$\min_{x\in\mathcal{X}} \; g_0(x) \quad \text{s.t.} \; g_j(x) \le 0$$

Composite Optimization $\longrightarrow$ $$\min_{x\in\mathcal{X}} \; g_0(x) + h(g_1(x), \dots, g_m(x))$$

From Constraints to Compositions

Constrained Optimization $\longrightarrow$ $\min_{x \in \mathcal{X}} g_0(x) + \iota_{z \le 0}(g_1(x), \ldots, g_m(x))$

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$$p_\star = \begin{cases} \min_{x\in\mathcal{X}} & g_0(x) \\ \text{s.t.} & g_j(x) \le 0 \end{cases}$$

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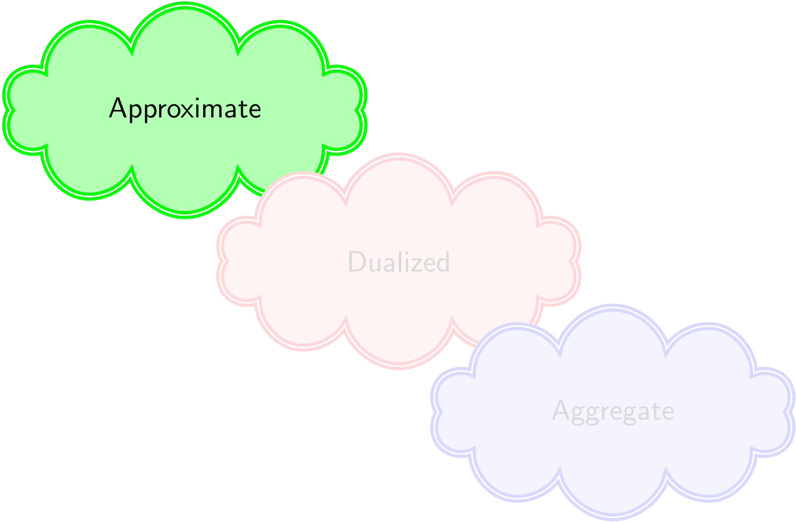

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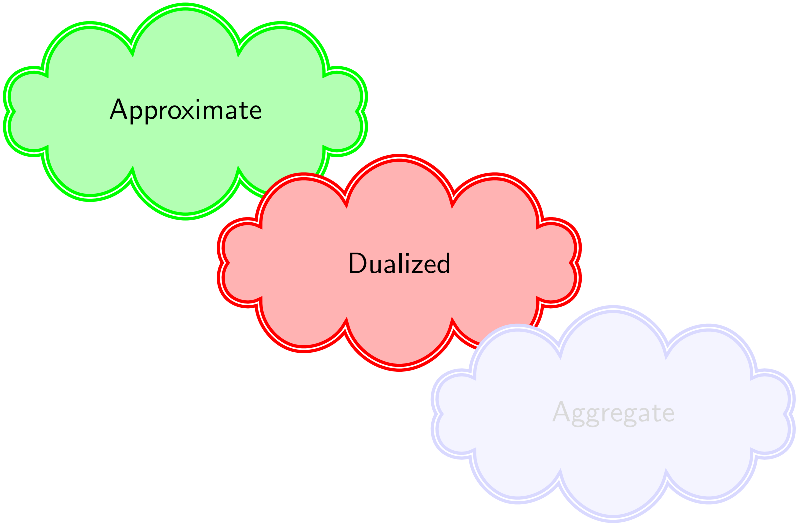

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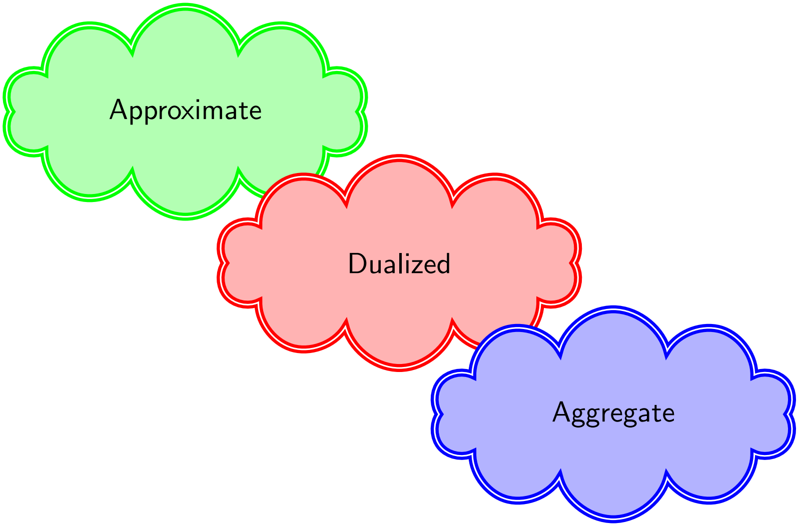

Tool 1: Fenchel Conjugates

Dualized

$$f^*(\lambda) = \sup_{x \in \mathbb{R}^n} \langle \lambda, x \rangle - f(x)$$

$$\Big\downarrow\; f^{**} = f$$

$$f(x) = \sup_{\lambda \in \mathbb{R}^n} \langle x, \lambda \rangle - f^*(\lambda)$$

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$$h^*(\lambda) = \sup_{x \in \mathbb{R}^n} \langle \lambda, x \rangle - h(x)$$

$$h^{**} = h$$

$$h(g(x)) = \sup_{\lambda \in \mathbb{R}^n} \langle \lambda, g(x) \rangle - h^*(\lambda)$$

Lagrangian Reformulation (via conjugates)

$$p_\star = \min_{x\in\mathcal{X}} g_0(x) + \boxed{\iota_{z\le 0}(g_1(x), \dots, g_m(x))}$$

$$p_\star = \min_{x\in\mathcal{X}} g_0(x) + \boxed{\max_{\lambda\in\mathbb{R}^m} \sum_{j=1}^{m} \lambda_j g_j(x) - \iota^*_{z\le 0}(\lambda)}$$

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Sums are Wonderful

Aggregate

$$\min_{x \in \mathcal{X}} \mathcal{L}(x, \lambda^\star) := g_0(x) + \sum_{j=1}^{m} \lambda_j^\star g_j(x)$$

each g_j is L_j-smooth

$$\sum_{j=0}^{m} \lambda_j^\star g_j(x) \text{ is } \sum_{j=0}^{m} \lambda_j^\star L_j\text{-smooth}$$

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Inaccessible $\lambda^\star$!

Approximate

$$\min_{x \in \mathcal{X}} \mathcal{L}(x, \lambda) := g_0(x) + \sum_{j=1}^{m} \lambda_j g_j(x)$$

restrict $\lambda \in \Lambda_r := B(\lambda^\star, r)$

$$\sum_{j=0}^{m} \lambda_j g_j(x) \text{ is } \sum_{j=0}^{m} (\lambda_j^\star + r) L_j\text{-smooth}$$

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Defining Our Constants

Approximate Dualized Aggregate Smoothness Constant I

When g_j is L_j-smooth

WLOG let $g_0(x) = 0$

$$\boxed{L^{\mathtt{ADA}}_{\varepsilon,r} := \sum_{j=1}^{m} (\lambda_j^\star + r) L_j}$$

Tool 2: Nesterov-Style Quadratic Upper Bounds

Lemma (Lemma 1, Nesterov [1])

Fix $\delta > 0$ and (L, p)-Hölder smooth, with $L_\delta \geq \left[\frac{1-p}{1+p}\frac{1}{\delta}\right]^{\frac{1-p}{1+p}} L^{\frac{2}{1+p}}$,

$$f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \frac{L_\delta}{2}\|y - x\|^2 + \frac{\delta}{2}, \ \forall x, y \in dom\ f \ . \tag{1}$$

Approximate Dualized Aggregate Smoothness Constant II

When g_j is (L_j, p_j)-Hölder smooth

setting $\delta = \frac{\varepsilon}{\sqrt{16 L^{\mathrm{ADA}}_{\varepsilon,r} D_x^2/\varepsilon}}$

$$L_{\delta,r} := \sum_{j=1}^{m} \left[\left[\frac{1-p_j}{1+p_j} \cdot \frac{m}{\delta} \right]^{\frac{1-p_j}{1+p_j}} \left[(\lambda_j^\star + r) \cdot L_j \right]^{\frac{2}{1+p_j}} \right]$$

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Tool 3: Restarting Methods

Suppose
we're given

Guarantees that
$f(x^N) - f(x^\star) \leq \frac{L\|x^0 - x^\star\|^2}{2N^2}$

Our algorithm produces output
s.t. $f(x^k) - f(x^\star) \leq \varepsilon$

$f(x^k) - f(x^\star) \geq \frac{\mu}{2}\|x^k - x^\star\|^2$

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How many iterations to reach $\varepsilon/2$?

Recall we have
$$\|x^k - x^\star\|^2 \le \frac{2\varepsilon}{\mu}$$

After restarting
$$f(x^N) - f(x^\star) \le \frac{L\|x^k - x^\star\|^2}{2N^2}$$

$N = \sqrt{2L/\mu}$ yields
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General Growth Condition

Suppose $f(x) = \sum_{j=1}^{m} \lambda_j^\star g_j(x)$

Each g_j is (μ_j, q_j)-uniformly convex

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General Growth Condition

Suppose $f(x) \;=\; \sum_{j=1}^{m} \lambda_j^\star g_j(x)$

$$\Big\Downarrow \text{ Each } g_j \text{ is } (\mu_j, q_j)\text{-uniformly convex}$$

$$f(x) - f(x^\star) \geq \underbrace{\sum_{j=1}^{m} \lambda_j^\star \frac{\mu_j}{q_j+1} \|x - x^\star\|^{q_j+1}}_{G_x(\|x - x^\star\|)}$$

Revisiting Restarting

Suppose we're given

Guarantees that
$f(x^N) - f(x^\star) \leq \frac{L^{\mathtt{ADA}}_{\varepsilon,r}\|x^0 - x^\star\|^2}{2N^2}$

Our algorithm produces output s.t. $f(x^k) - f(x^\star) \leq \varepsilon$

$f(x^k) - f(x^\star) \geq G_x\left(\|x^k - x^\star\|\right)$

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$$f(x^k) - f(x^\star) \geq G_x\left(\|x^k - x^\star\|\right)$$

Revisiting Restarting

How many iterations to reach $\varepsilon/2$?

Recall we have $\|x^k - x^\star\|^2 \leq (G_x^{-1}(\varepsilon))^2$

After restarting $f(x^N) - f(x^\star) \leq \frac{L^{\mathrm{ADA}}_{\varepsilon,r}\|x^k - x^\star\|^2}{2N^2}$

$N = \sqrt{\frac{2L^{\mathrm{ADA}}_{\varepsilon,r}(G_x^{-1}(\varepsilon))^2}{2\varepsilon}}$ yields $f(x^N) - f(x^\star) \leq \frac{\varepsilon}{2}$

Revisiting Restarting

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$\mu^{\mathtt{ADA}}_{\varepsilon}$?

Revisiting Restarting

How many iterations to reach $\varepsilon/2$?

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$$\|x^k - x^\star\|^2 \leq (G_x^{-1}(\varepsilon))^2$$

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$$f(x^N) - f(x^\star) \leq \frac{L^{\mathtt{ADA}}_{\varepsilon,r}\|x^k - x^\star\|^2}{2N^2}$$

$$N = \sqrt{\frac{2L^{\mathtt{ADA}}_{\varepsilon,r}(G_x^{-1}(\varepsilon))^2}{2\varepsilon}}$$
yields $f(x^N) - f(x^\star) \leq \frac{\varepsilon}{2}$

$1/\mu^{\mathtt{ADA}}_{2\varepsilon}$

Approximate Dualized Aggregate Convexity Constant

When g_j is (μ_j, q_j)-uniformly convex

$$\mu_\varepsilon^{\texttt{ADA}} = \frac{\varepsilon}{(G_x^{-1}(\varepsilon/2))^2}$$

Approximate Dualized Aggregate Convexity Constant

When g_j is (μ_j, q_j)-uniformly convex

$$\mu_\varepsilon^{\mathtt{ADA}} := \left\{ \mu^{\mathtt{ADA}} > 0 : \frac{\mu^{\mathtt{ADA}}}{2} = \sum_{j=1}^{m} \lambda_j^\star \frac{\mu_j}{q_j+1} (\varepsilon/\mu^{\mathtt{ADA}})^{\frac{q_j-1}{2}} \right\}$$

Approximate Dualized Aggregate Smoothness Constant III

When g_j is (L_j, p_j)-Hölder smooth and (μ_j, q_j)-uniformly convex

$$L^{\text{ADA}}_{\varepsilon,r} := \left\{ L^{\text{ADA}} > 0 : L^{\text{ADA}} = \sum_{j=1}^{m} \left[\frac{1-p_j}{1+p_j} \cdot \frac{m\sqrt{L^{\text{ADA}}}}{\varepsilon} \cdot \min\left\{ \frac{4D_x}{\sqrt{\varepsilon}}, \frac{8}{\sqrt{\mu^{\text{ADA}}_{\varepsilon}}} \right\} \right]^{\frac{1-p_j}{1+p_j}} \left[(\lambda_j^\star + r) L_j \right]^{\frac{2}{1+p_j}} \right\}$$

Algorithms and Analysis

Extended Lagrangian and Gap Function

$$F(x) = h(\underbrace{g_1(x), \ldots, g_m(x)}_{g(x)}) + u(x)$$

$$\mathcal{L}(x; \lambda) := \langle \lambda, g(x) \rangle - h^*(\lambda) + u(x)$$

Extended Lagrangian and Gap Function

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Extended Lagrangian and Gap Function

$$\mathcal{L}(x;\lambda) := \langle \lambda, g(x) \rangle - h^*(\lambda) + u(x)$$

$$\mathcal{L}(x;\lambda,\nu) := \langle \lambda, \underbrace{\nu x - g^*(\nu)}_{\text{conjugate of g}} \rangle - h^*(\lambda) + u(x)$$

Extended Lagrangian and Gap Function

$$\mathcal{L}(x;\lambda,\nu) := \langle \lambda, \underbrace{\nu x - g^*(\nu)}_{\text{conjugate of g}} \rangle - h^*(\lambda) + u(x)$$

Gap Function

$$Q(z,\hat{z}) = \mathcal{L}(x;\hat{\lambda},\hat{\nu}) - \mathcal{L}(\hat{x};\lambda,\nu)$$

Minimizing the Gap Function ("Q-Analysis" and Sliding)

$$Q(z^t; z) = Q_\nu(z^t; z) + Q_\lambda(z^t; z) + Q_x(z^t; z)$$

$$\begin{aligned} &\mathcal{L}(x^t; \lambda, \nu) - \mathcal{L}(x^t; \lambda, \nu^t) \\ = \quad & \langle \lambda, \nu x^t - g^*(\nu) \rangle \boxed{-\langle \lambda, \nu^t x^t - g^*(\nu^t) \rangle} \end{aligned}$$

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$$\begin{aligned} &\mathcal{L}(x^t; \lambda^t, \nu^t) - \mathcal{L}(x; \lambda^t, \nu^t) \\ = \quad & \boxed{\langle \sum_{i=1}^m \lambda_i^t \nu_i^t, x^t \rangle + u(x^t)} - \langle \textstyle\sum_{i=1}^m \lambda_i^t \nu_i^t, x \rangle - f(x) \end{aligned}$$

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$$\mathcal{L}(x^t; \lambda^t, \nu^t) - \mathcal{L}(x; \lambda^t, \nu^t)$$
$$= \boxed{\langle \sum_{i=1}^{m} \lambda_i^t \nu_i^t, x^t\rangle + u(x^t)} - \langle \textstyle\sum_{i=1}^m \lambda_i^t \nu_i^t, x\rangle - f(x)$$

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$$Q(z^t; z) \quad = \quad Q_\nu(z^t; z) \; + \; Q_\lambda(z^t; z) \; + \; Q_x(z^t; z)$$

$$\nu_j^t \;\; \leftarrow \text{argmax}_{\nu_j \in V_j} \langle \nu, \tilde{x}^t \rangle \; - \; g_j^*(\nu) \; - \; \tau_t U_{g_j^*}(\nu_j; \nu_j^{t-1})$$

$$\begin{aligned} & \mathcal{L}(x^t; \lambda, \nu^t) - \mathcal{L}(x^t; \lambda^t, \nu^t) \\ = \; & \langle \lambda, \nu^t x^t - g^*(\nu^t) \rangle \; - \; h^*(\lambda) \boxed{- \left[\langle \lambda^t, \nu^t x^t - g^*(\nu^t) \rangle - h^*(\lambda^t) \right]} \end{aligned}$$

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$$\lambda^t \leftarrow \text{argmax}_{\lambda \in \Lambda} \langle \lambda, \nu^t \tilde{x}^t - g^*(\nu^t) \rangle - h^*(\lambda) - \tfrac{\gamma_t}{2} \|\lambda - \lambda^{t-1}\|^2$$

$$\begin{aligned} & \mathcal{L}(x^t; \lambda^t, \nu^t) - \mathcal{L}(x; \lambda^t, \nu^t) \\ = \; & \boxed{\langle \sum_{i=1}^{m} \lambda_i^t \nu_i^t, x^t \rangle + u(x^t)} \; - \; \langle \textstyle\sum_{i=1}^{m} \lambda_i^t \nu_i^t, x \rangle \; - \; f(x) \end{aligned}$$

Minimizing the Gap Function ("Q-Analysis" and Sliding)

$$Q(z^t; z) \;=\; Q_\nu(z^t; z) + Q_\lambda(z^t; z) + Q_x(z^t; z)$$

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$$x^t \leftarrow \mathrm{argmin}_{x \in \mathcal{X}} \langle \textstyle\sum_{j=1}^m \lambda_j^t \nu_j^t, x \rangle + u(x) + \tfrac{\eta_t}{2} \|x - x^{t-1}\|^2$$

Minimizing the Gap Function ("Q-Analysis" and Sliding)

$$Q(z^t; z) \quad = \quad Q_\nu(z^t; z) + Q_\lambda(z^t; z) + Q_x(z^t; z)$$

$$\nu^t \leftarrow \nabla g(\underline{x}^t),\ \underline{x}^t \leftarrow \tfrac{\tau_t \underline{x}^{t-1} + \tilde{x}^t}{1+\tau_t} \text{ with } \tilde{x}^t = x^{t-1} + \theta_t(x^{t-1} - x^{t-2})$$

$$\lambda^t \leftarrow \text{argmax}_{\lambda \in \Lambda} \langle \lambda, \nu^t \tilde{x}^t - g^*(\nu^t) \rangle - h^*(\lambda) - \tfrac{\gamma_t}{2} \|\lambda - \lambda^{t-1}\|^2$$

$$x^t \leftarrow \text{argmin}_{x \in \mathcal{X}} \langle \textstyle\sum_{j=1}^m \lambda_j^t \nu_j^t, x \rangle + u(x) + \tfrac{\eta_t}{2} \|x - x^{t-1}\|^2$$

The Universal Fast Composite Method (UFCM)

Algorithm 1 Universal Fast Composite Method (UFCM)

Input $z^0 \in \mathcal{X} \times \Lambda$, outer loop iteration count T, and smoothness constant $L^{\mathsf{ADA}}_{\varepsilon,r}$

Initialize $x^{-1} = \underline{x}^0 = y_0^{(1)} = x^0 \in \mathcal{X}$, $\lambda_{-1}^{(1)} = \lambda_0^{(1)} = \lambda^0 \in \Lambda$, and parameters $\{\theta_t\}, \{\eta_t\}, \{\tau_t\}, \{\omega_t\}$ as a function of $L^{\mathsf{ADA}}_{\varepsilon,r}$

Set $\nu^0 = \nabla g(x^0)$.
for $t = 1,\ 2,\ 3,\ ...,\ T$ **do**
  Set $\underline{x}^t \leftarrow (\tau_t \underline{x}^{t-1} + \tilde{x}^t)/(1+\tau_t)$ where $\tilde{x}^t = x^{t-1} + \theta_t(x^{t-1} - x^{t-2})$
  Set $\nu^t \leftarrow \nabla g(\underline{x}^t)$
  Calculate inner loop iteration limit S_t, paramters $\beta^{(t)}$, $\gamma^{(t)}$, and $\rho^{(t)}$
  for $s = 1, 2, ..., S_t$ **do**
    Set $\tilde{h}^{(t),s} = \begin{cases} (\nu^t)^T \lambda_0^{(t)} + \rho^{(t)} (\nu^{t-1})^T (\lambda_0^{(t)} - \lambda_{-1}^{(t)}) & \text{if } s = 1, \\ (\nu^t)^T \lambda_{s-1}^{(t)} + (\nu^t)^T (\lambda_{s-1}^{(t)} - \lambda_{s-2}^{(t)}) & \text{otherwise} \end{cases}$
    Solve $y_s^{(t)} \leftarrow \operatorname*{argmin}_{y \in \mathcal{X}} \left\langle \tilde{h}^{(t),s}, y \right\rangle + u(y) + \frac{\eta_t}{2}\|y - x^{t-1}\|^2 + \frac{\beta^{(t)}}{2}\|y - y_{s-1}^{(t)}\|^2$
    Solve $\lambda_s^{(t)} \leftarrow \operatorname*{argmax}_{\lambda \in \Lambda} \left\langle \lambda, \nu^t (y_s^{(t)} - \underline{x}^t) + g(\underline{x}^t) \right\rangle - h^*(\lambda) - \frac{\gamma^{(t)}}{2}\|\lambda - \lambda_{s-1}^{(t)}\|^2$
  end for
  Set $\lambda_0^{(t+1)} = \lambda_{S_t}^{(t)}$, $\lambda_{-1}^{(t+1)} = \lambda_{S_t-1}^{(t)}$, $y_0^{(t+1)} = y_{S_t}^{(t)}$
  Set $x^t = \sum_{s=1}^{S_t} y_s^{(t)}/S_t$ and $\tilde{\lambda}^t = \sum_{s=1}^{S_t} \lambda_s^{(t)}/S_t$
end for
return $(\bar{x}^T, \bar{\lambda}^T) := \sum_{t=1}^{T} \omega_t \left(x^t, \tilde{\lambda}^t\right) / \left(\sum_{t=1}^{T} \omega_t\right)$

Figure: Modified from [3]

Restarted-UFCM

Algorithm 2 Restarted Universal Fast Composite Method (R-UFCM)

Input $z^0 \in \mathcal{X} \times \Lambda$, distance bounds D_x and D_λ, target accuracy $\varepsilon > 0$, constants $L^{\mathsf{ADA}}_{\varepsilon,r}$ and $\mu^{\mathsf{ADA}}_{\varepsilon}$, and UFCM execution count $K = \left\lceil \log_2\left(\frac{Q(\bar{z}^0,\hat{z})+\varepsilon}{\varepsilon}\right)\right\rceil$

Set $D_x^{(0)}$, $D_\lambda^{(0)}$ and $\{T_k\}$ according to (5.3)

for $k = 0, 1, \ldots, K-1$ **do**

  Run UFCM(z^k, $\lceil T_k \rceil$, $L^{\mathsf{ADA}}_{\varepsilon,r}$) returning output $(\bar{x}^{T_k,k}, \bar{\lambda}^{T_k,k})$

  Set $(x^{k+1}, D_x^{(k+1)}) = \begin{cases} (\bar{x}^{T_k,k}, \sqrt{2^{K-k}\varepsilon/\mu^{\mathsf{ADA}}_{\varepsilon}}) & \text{if } \mu^{\mathsf{ADA}}_{\varepsilon} \geq 4\varepsilon/D_x^2 \\ (x^0, D_x) & \text{otherwise} \end{cases}$

  Set $(\lambda^{k+1}, D_\lambda^{(k+1)}) = \begin{cases} (\bar{\lambda}^{T_k,k}, \sqrt{2^{K-k}\varepsilon L_h}) & \text{if } L_h \leq D_\lambda^2/\varepsilon \\ (\lambda^0, D_\lambda) & \text{otherwise} \end{cases}$

  Set $z^{k+1} = (x^{k+1}, \lambda^{k+1})$

end for

Figure: Restarted Variant

Universally Optimal Guarantees

Theorem 1: $\mathcal{O}\left(\sqrt{\frac{L^{\text{ADA}}_{\varepsilon,r} D_x^2}{\mu^{\text{ADA}}_{\varepsilon}}}\right)$

total gradient calls to g

when the objective

is sufficiently convex

$(\mu^{\text{ADA}}_{\varepsilon} \geq \varepsilon / D_x^2)$

Theorem 2: $\mathcal{O}\left(\sqrt{\frac{L^{\text{ADA}}_{\varepsilon,r}}{\mu^{\text{ADA}}_{\varepsilon}}} \log\left(\frac{1}{\varepsilon}\right)\right)$

total gradient calls to g

Universally Optimal Guarantees

Theorem 1: $\mathcal{O}\left(\sqrt{\frac{L^{\mathtt{ADA}}_{\varepsilon,r} D_x^2}{\mu^{\mathtt{ADA}}_{\varepsilon}}}\right)$ total gradient calls to g

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Theorem 2: $\mathcal{O}\left(\sqrt{\frac{L^{\mathtt{ADA}}_{\varepsilon,r}}{\mu^{\mathtt{ADA}}_{\varepsilon}}} \log\left(\frac{1}{\varepsilon}\right)\right)$ total gradient calls to g

Immediate Corollaries

Suppose g is (L, p)-Hölder smooth

$$\mathcal{O}\left(\left(\frac{L}{\varepsilon}\right)^{\frac{2}{1+3p}}\|x^0 - x^\star\|^{\frac{2+2p}{1+3p}}\right)$$

total gradient calls to g

Immediate Corollaries

Suppose g is (L, p)-Hölder smooth

$$\mathcal{O}\left(\left(\frac{L}{\varepsilon}\right)^{\frac{2}{1+3p}} \left\|x^0 - x^\star\right\|^{\frac{2+2p}{1+3p}}\right)$$

total gradient calls to g

Immediate Corollaries

$$L^{\texttt{ADA}}_{\varepsilon,r} = (1+r)^{\frac{4}{1+3p}} \left[\frac{1-p}{1+p} \cdot \frac{4D_x}{\varepsilon\sqrt{\varepsilon}}\right]^{\frac{2-2p}{1+3p}} L^{\frac{4}{1+3p}}$$

$$\begin{aligned} &\mathcal{O}\left(\sqrt{\frac{L^{\texttt{ADA}}_{\varepsilon,r} D_x^2}{\varepsilon}}\right) \\ &= \mathcal{O}\left(\sqrt{\left(\frac{D_x}{\varepsilon\sqrt{\varepsilon}}\right)^{\frac{2-2p}{1+3p}} \frac{L^{\frac{4}{1+3p}} D_x^2}{\varepsilon}}\right) \\ &= \mathcal{O}\left(\left(\frac{L}{\varepsilon}\right)^{\frac{2}{1+3p}} D_x^{\frac{2+2p}{1+3p}}\right) \end{aligned}$$

Immediate Corollaries

$$L_{\varepsilon,r}^{\mathtt{ADA}} = (1+r)^{\frac{4}{1+3p}} \left[\frac{1-p}{1+p} \cdot \frac{4D_x}{\varepsilon\sqrt{\varepsilon}}\right]^{\frac{2-2p}{1+3p}} L^{\frac{4}{1+3p}}$$

$$\begin{aligned} &\mathcal{O}\left(\sqrt{\frac{L_{\varepsilon,r}^{\mathtt{ADA}} D_x^2}{\varepsilon}}\right) \\ &= \mathcal{O}\left(\sqrt{\left(\frac{D_x}{\varepsilon\sqrt{\varepsilon}}\right)^{\frac{2-2p}{1+3p}} \frac{L^{\frac{4}{1+3p}} D_x^2}{\varepsilon}}\right) \\ &= \mathcal{O}\left(\left(\frac{L}{\varepsilon}\right)^{\frac{2}{1+3p}} D_x^{\frac{2+2p}{1+3p}}\right) \end{aligned}$$

Immediate Corollaries II

Suppose g is (L,p)-Hölder smooth *and* (μ,q)-uniformly convex

$$\begin{cases} \mathcal{O}\left(\left(\frac{L^{1+q}}{\mu^{1+p}\varepsilon^{q-p}}\right)^{\frac{2}{(1+3p)(1+q)}}\right) & \text{if } q>p\ , \\ \mathcal{O}\left(\left(\frac{L^{1+q}}{\mu^{1+p}}\right)^{\frac{2}{(1+q)(1+3p)}}\log\left(\frac{G(x^0)-G^\star}{\varepsilon}\right)\right) & \text{if } q=p \end{cases}$$

total gradient calls to g

Immediate Corollaries II

Suppose g is (L, p)-Hölder smooth *and* (μ, q)-uniformly convex

$$\begin{cases} \mathcal{O}\left(\left(\frac{L^{1+q}}{\mu^{1+p}\varepsilon^{q-p}}\right)^{\frac{2}{(1+3p)(1+q)}}\right) & \text{if } q > p\ , \\ \mathcal{O}\left(\left(\frac{L^{1+q}}{\mu^{1+p}}\right)^{\frac{2}{(1+q)(1+3p)}} \log\left(\frac{G(x^0)-G^\star}{\varepsilon}\right)\right) & \text{if } q = p \end{cases}$$

total gradient calls to g

Immediate Corollaries II

$$L^{\mathtt{ADA}}_{\varepsilon,r} = \left[\frac{1-p}{1+p} \cdot \frac{8}{\varepsilon\sqrt{\mu^{\mathtt{ADA}}_{\varepsilon}}}\right]^{\frac{2-2p}{1+3p}} L^{\frac{4}{1+3p}}$$

$$\mu^{\mathtt{ADA}}_{\varepsilon} = 2\left(\frac{\mu}{1+q}\right)^{\frac{2}{1+q}} \varepsilon^{\frac{q-1}{q+1}}$$

$$\tilde{O}\left(\sqrt{\frac{L^{\mathtt{ADA}}_{\varepsilon,r}}{\mu^{\mathtt{ADA}}_{\varepsilon}}}\right) = \tilde{O}\left(\frac{\left(\varepsilon\sqrt{\mu^{\mathtt{ADA}}_{\varepsilon}}\right)^{-\frac{1-p}{1+3p}} L^{\frac{2}{1+3p}}}{\sqrt{\mu^{\mathtt{ADA}}_{\varepsilon}}}\right) = \tilde{O}\left(\left(\frac{L^{1+q}}{\mu^{1+p}\varepsilon^{q-p}}\right)^{\frac{2}{(1+3p)(1+q)}}\right),$$

Immediate Corollaries II

$$L_{\varepsilon,r}^{\texttt{ADA}} = \left[\frac{1-p}{1+p} \cdot \frac{8}{\varepsilon\sqrt{\mu_\varepsilon^{\texttt{ADA}}}}\right]^{\frac{2-2p}{1+3p}} L^{\frac{4}{1+3p}}$$

$$\mu_\varepsilon^{\texttt{ADA}} = 2\left(\frac{\mu}{1+q}\right)^{\frac{2}{1+q}} \varepsilon^{\frac{q-1}{q+1}}$$

$$\begin{aligned} &\tilde{O}\left(\sqrt{\frac{L_{\varepsilon,r}^{\texttt{ADA}}}{\mu_\varepsilon^{\texttt{ADA}}}}\right) \\ &= \tilde{O}\left(\frac{\left(\varepsilon\sqrt{\mu_\varepsilon^{\texttt{ADA}}}\right)^{-\frac{1-p}{1+3p}} L^{\frac{2}{1+3p}}}{\sqrt{\mu_\varepsilon^{\texttt{ADA}}}}\right) \\ &= \tilde{O}\left(\left(\frac{L^{1+q}}{\mu^{1+p}\varepsilon^{q-p}}\right)^{\frac{2}{(1+3p)(1+q)}}\right), \end{aligned}$$

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