Linear Algebra Review Session Day 1 (part 1)

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Outline

- Motivation for Linear Algebra

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- Fields ($\mathbb{F}$), Lists ($\mathbb{F}^n$), Operations in $\mathbb{F}^n$

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- Inner Products $\rightarrow$ Norms $\rightarrow$ Metrics Spaces
- Gram-Schmidt Orthogonalization

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- Why is one not *easy*?
- Why are these both equally *easy*?

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- This is the essence of math... Math is hard, nearly impossible without the right tools.
- Linear Algebra helps use create tools to answer tricky problems!

Why Linear Algebra?

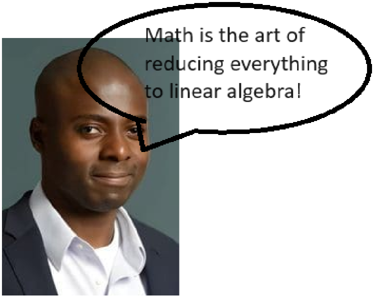

Figure: Dr. Edinah Grang

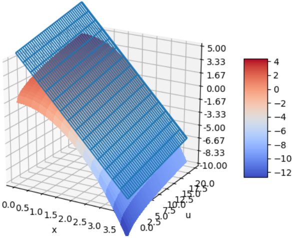

Figure: Linearization

Fields

In order to do linear algebra, we need numbers. Or at the very least, we need stuff to work with–something that looks familiar (to the real numbers) but is general enough to handle more *complex* settings

As we'll see, the *linearity* of *linear* algebra, needs us to be able to add, scale (multiply by a constant), and distribute the two operations. Thus, we need an ambient space to do these calculations.

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6. Distributivity: $\forall\ a, b, c \in F,\ a \cdot (b + c) = a \cdot b + a \cdot c$

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- What if we want to handle multiple values at once? Say we want to keep track of position with an x, y, and z coordinate.

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List: For a fixed natural number $n \in \mathbb{N}$, define an *n-dimensional list over field* $\mathbb{F}$ to be

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Remark

We call x_j then j^{th}-coordinate

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- **(additive) Inverse**: need for $\mathbf{x} + -\mathbf{x} = \mathbf{0}$, so we define $-\mathbf{x} = (-x_1, ..., -x_n)$

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Remark

We immediately get associativity, commutativity, and distributivity (of scalar multiplication) from the properties of the underlying field. We also get uniques of inverses, identities, and the properties that $\mathbf{x} = 1 \cdot \mathbf{x}$ *and* $-\mathbf{x} = (-1) \cdot \mathbf{x}$

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Proof.

$$\mathbf{x} + \mathbf{y} = (x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n) = (y_1 + x_1, ..., y_n + x_n) = (y_1, ..., y_n) + (x_1, ..., x_n) = \mathbf{y} + \mathbf{x}$$

□

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6. $\begin{cases} a(u + v) = au + av & \forall a \in \mathbb{F},\ u, v \in V \\ (a + b)v = av + bv & \forall a, b \in \mathbb{F},\ v \in V \end{cases}$

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Remark (notation)

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- *and* $a \cdot v = av$ *going forward.*

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Remark

$\mathbb{F}^n$ and $\mathbb{F}^\infty$ are of the latter form, for $S = \{1, 2, ..., n\}$ with $n \in \mathbb{N} \cup \{\infty\}$ as $(x_1, ...x_n)$ can be see as the output of the function $f : \{1, ..., n\} \to \mathbb{F}$ with $x_j = f(j)$

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- In summary, there are many types of vector spaces, but the easiest to understand and utilize are *finite dimensional* vector spaces (will see soon).

Some Properties of Vector Spaces

$$\begin{cases} 0v = v \ \forall v \in V & Proof: \cancel{0v} = (0+0)v = \cancel{0v} + 0v \implies 0 = 0v \\ a0 = 0 \ \forall a \in \mathbb{F} & Proof: \cancel{a0} = a(0+0) = \cancel{a0} + a0 \implies a0 = 0 \\ (-1)v = -v \ \forall v \in V & Proof: (-1)v = 0 + (-1)v = (-v+v) + (-1)v \\ & = (-v) + ((1+-1)v) = (-v) \\ \text{Unique Identity} & Proof: 0' = 0' + 0 = 0 \\ \text{Unique Inverses} & Proof: w = w + 0 = w + (v + w') \\ & = (w+v) + w' = 0 + w' = w' \\ \sum_{i=1}^{n} v_i \in V, v_i \in V & Proof: \text{By induction...} \underbrace{(v_1 + ... v_{n-1})}_{\in V} + v_n \in V \end{cases}$$

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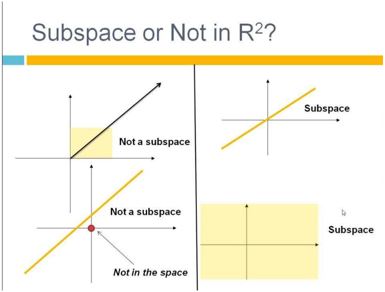
Remark

We actually just have to check the first because the other two are direct corollaries! Suppose 1) holds, then:

$$0 = 0 * v \in V \quad -v = (-1)v \in V$$

Subspace (examples)

- $U_1 = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{F}\}$ is a subspace of $\mathbb{F}^3$



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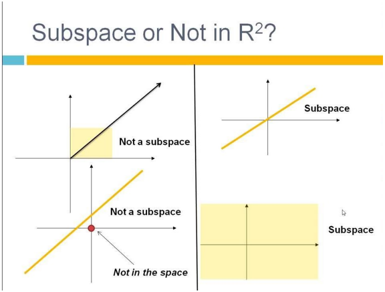

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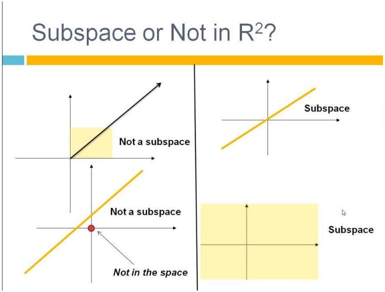

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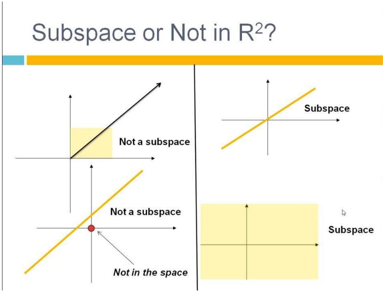

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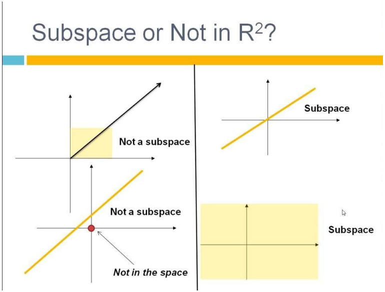

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For two subspaces $U_1, U_2 \subseteq V$, we denote the sum $U_1 + U_2$ as

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Direct Sum

Definition

We say a sum of subspace $U_1 + ... + U_m$ is a **direct sum** iff each element $v = u_1 + ... + u_m,\ u_j \in U_j$ has *unique* representation
We write this as $U_1 \oplus ... \oplus U_m$

That is to say, if

$$v = u_1 + ... + u_m$$
$$v = \hat{u}_1 + ... + \hat{u}_m$$

Then $u_j = \hat{u}_j$ for all $j = 1, ..., m$
Equivalently, by subtracting both equations, we get as necessary and sufficient condition for a sum of subspaces being a **direct sum**:

$$0 = u_1 + ... + u_m \iff u_1 = u_2 = ... = u_m = 0$$

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- We can add subspaces in the same way we add sets, and by the closure of the subspaces, the sum is also a subspace
- Direct sums are the "disjoint subset" analog, and in the case of adding two subspaces, we have direct sum iff $U_1 \cap U_2 = \{0\}$