Linear Algebra Review Session Day 1 (part 2)

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Span and Linear Independence

Definition

Given a (finite) set of vectors $\{v_1, ..., v_m\}$, a **linear combination** is of the form

$$a_1 v_1 + ... + a_m v_m = \sum_{i=1}^{m} a_i v_i$$

for $a_i \in \mathbb{F}$ Thus, a linear combination is a function $f : \underbrace{V \times ... \times V}_{m \ \ times} \to V$ that is linear in each argument (we will see more of this later...)

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- *If both $a_i \geq 0$ and $\sum_{i=1}^m a_i = 1$, then this is called a* **convex** *combination*

Span

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The **span** of a subset $S \subseteq V$ is all the collected of (finite) linear combinations:

$$\text{span}(S) = \left\{ \sum_{i=1}^{k} a_i v_i | k \in \mathbb{N}, v_i \in S, a_i \in \mathbb{F} \right\}$$

We say S *spans* $U \subseteq V$ if every element $u \in U$ can be written as a linear combination in S

Example:

$$\{\underbrace{(1, 0, 0, ..., 0)}_{e_1}, \underbrace{(0, 1, 0, ..., 0)}_{e_2}, \underbrace{(0, 0, 1, ..., 0)}_{e_3}, ..., \underbrace{(0, 0, 0, ..., 1)}_{e_n}\} \text{ spans } \mathbb{F}^n$$

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Remark

We can think about the span(S) as the smallest subspace that contains S. Similarly, $U_1 + U_2$is the smallest subspace containing U_1 and U_2

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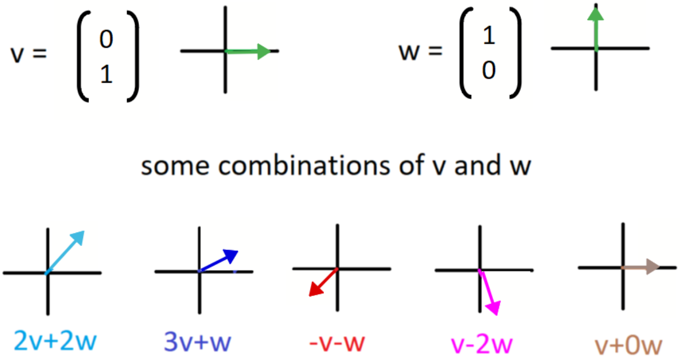

Figure: Medium

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We saw V is **finite dimensional** if there exist a finite set $S \subseteq V$ (here this is just a subset, not a subspace) where

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- $\mathcal{P}(\mathbb{R})$ *is not finite dimensional, because any finite list of polynomials (say up to degree k) will not have* x^{k+1} *in their span*
  - *We can then say that* x^{k+1} *is* **linearly independent** *(cannot be written as a linear combination) from the elements* $1, x, x^2, ..., x^k$

Linear Independence

On the last slide, we said that the "vector" x^{k+1} is linearly independent from $\{1, x, x^2, ..., x^k\}$ because it cannot be written as a linear combination of them. He we give the formal definition:

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A set of vectors $\{v_1, ..., v_m\}$ is called **linearly independent** if

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3. In other words, for some $i = 1, .., m$:

$$v_i \notin \text{span}(\{v_1, ..., v_m\} \backslash v_i)$$

Linear Independence

1.

($\Longleftarrow$)If each $a_i = 0$, then $\sum_{i=1}^m a_i v_i = \sum_{i=1}^m 0 v_i = 0$
(Uniqueness) Suppose the linear independence property holds. Then if we have two representations of $v \in V$

$$v = \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i v_i$$

Then we get

$$\sum_{i=1}^m (a_i - b_i) v_i = 0 \Longrightarrow a_i - b_i = 0 \ \forall i$$

So $a_i = b_i$ and the "representation" is the same. There is a unique way to write v as a linear combination of the $v_i's$

□

Linear Independence

2.

Letting $U_i = \text{span}(\{v_i\})$, then

$$
\begin{aligned}
& U_1 \oplus \ldots \oplus U_m \text{ is a direct sum} \\
\textit{iff} \quad & 0 = u_1 + \ldots + u_m,\ u_i \in U_i \iff u_i = 0\ \forall i \\
\textit{iff} \quad & 0 = a_1 v_1 + \ldots + a_m v_m, a_i \in \mathbb{F} \iff a_i = 0\ \forall i \\
\textit{iff} \quad & \{v_1, \ldots, v_m\} \text{ is linearly independent}
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- $a_iv_i = -(a_1v_1 + ... + a_{i-1}v_{i-1} + a_{i+1}v_{i+1} + ... + a_mv_m)$, with $a_i \neq 0$

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- $v_i = \frac{-a_1}{a_i} v_1 + ... + \frac{-a_{i-1}}{a_i} v_{i-1} + \frac{-a_{i+1}}{a_i} v_{i+1} + ... + \frac{-a_m}{a_i} v_m$, with $a_i \neq 0$

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- $v_i = b_1 v1 + ... + b_{i-1} v_{i-1} + b_{i+1} v_{i+1} ... + b_m v_m$ for $b_j \in \mathbb{F}$

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Remark

Note that if $\{v_1, ..., v_m\}$ is linearly dependent, we can remove one of the vectors (the one that can be written in the span of the other) to get a strictly smaller list with the same span.

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- Repeat until all the $u's$ are added. This took exactly m steps, and at least one w was removed each time. So if there are no $w's$ left over, then $m = n$. If there are left over $w's$ needs to span V then $m < n$

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- However, note that we will never have removed all the $w's$ before m steps because then that implies that $u_1, .., u_k$ spans V for some $k < m$, and so that contradicts the L.I. aspect

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Remark

Equivalently, this means that every vector $v \in V$ can be written uniquely as a linearly combination of v:

$$\forall v \in V \ \exists! \ a_1, \ldots, a_m \in \mathbb{F} \ \ s.t. \ \ v = a_1 v_1 + \ldots + a_m v_m$$

Spanning gives the existence
Linear independence gives the uniqueness

Examples of Bases

Examples of some bases of vectors spaces/subspaces

- $\{\underbrace{(1,0,0,...,0)}_{e_1},\underbrace{(0,1,0,...,0)}_{e_2},\underbrace{(0,0,1,...,0)}_{e_3},...,\underbrace{(0,0,0,...,1)}_{e_n}\}$ is called the *standard basis* of $\mathbb{F}^n$

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- $\{1,x,x^2,...,x^n\}$ is the standard basis of $\mathcal{P}_n(\mathbb{F})$

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  - Technically this just shows the span, but L.I. follows easily

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- *therefore* $|B_1| = |B_2|$

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Corollary

$$\dim U_1 \oplus U_2 = \dim U_1 + \dim U_2$$

Coordinates

Given a basis $\mathcal{B} = \{v_1, ..., v_m\}$ of vector space V, we can represent any vector v uniquely as

$$v = a_1 v_1 + ... + a_n v_n$$

Definition

We call these scalars **coordinates**, respect to the basis $\mathcal{B}$, and write

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Coordinates

The vector is whatever the vector is in the Vector Space. However, we can "name it" with a Basis. But just like different languages sound different, diffing Basis names will look different.

example.

For bases $\mathcal{B}_1 = \{(1,0),(0,1)\}$ and $\mathcal{B}_2 = \{(2,3),(3,-1)\}$ in $\mathbb{R}^2$, we can write the vector $v := (5,2)$ as

$$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_2}$$

□

Why do we care? Sometimes a vector "looks nicer" or is "easier to work with" in a new basis. We compromise *complexity of the basis* for *simplicity of the vectors we need to work with*.

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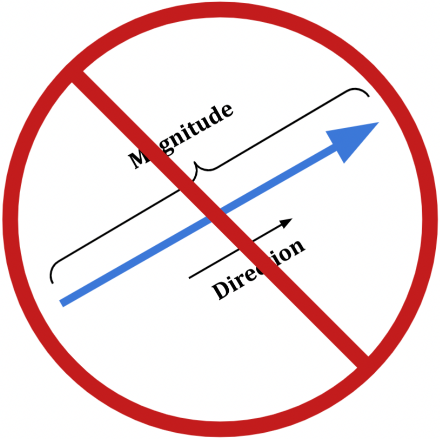
- Different **bases** have the same length–**dimension**. Why?
- Because changing our perspective doesn't change the amount of parameters *or information* is needed to locate something

What is a vector?

Figure: Despicable Me

What is a vector?



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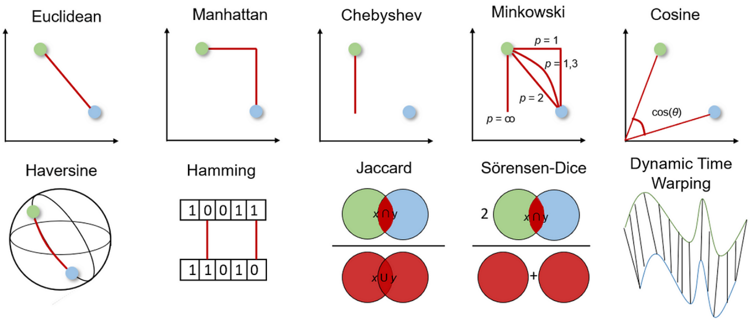
3. $d(u, w) \leq d(u, v) + d(v, w)$ (triangle inequality)

Visual Examples of Metrics



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- $d_H(u,v) = |\{j : u_j \neq v_j\}|$, number of components not equal (only for finite dimensional spaces)

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- For $f \in C[0,1]$, $\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{1/p}$

Inducing a metric from a norm

Note: given any norm $\|\cdot\| : V \to [0, \infty)$, we can *induce* a metric $d(u, v) : V \times V \to [0, \infty)$ by defining

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Proof.

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Remark

We also get that $\langle v, u + w \rangle = \langle v, u \rangle + \langle v, w \rangle$ *since*

$$\langle v, u + w \rangle = \overline{\langle u + w, v \rangle} = \overline{\langle u, v \rangle + \langle w, v \rangle} = \overline{\overline{\langle v, u \rangle} + \overline{\langle v, w \rangle}} = \langle v, u \rangle + \langle v, w \rangle$$

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Inner products are way more restrictive...

- $\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$ if $\mathbb{F} = \mathbb{R}$

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Given $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$, *define*

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Cauchy-Schwarz

In order to prove the triangle inequality, we need a detour...

Lemma (Cauchy-Schwarz)

For any $u, v \in V$ *with inner product* $\langle \cdot, \cdot \rangle$ *anmd induced norm* $\|v\| := \sqrt{\langle v, v \rangle}$, *then*

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Proof.

Consider just when $\mathbb{F} = \mathbb{R}$ for simplicity. Then for any $t \in \mathbb{R}$

$$\langle u + tv, u + tv \rangle \geq 0$$

and so $t^2 \underbrace{\|v\|^2}_{a} + t \underbrace{(2\langle u, v \rangle)}_{b} + \underbrace{\|u\|^2}_{c} \geq 0$ for all t (imagine a quadratic not crossing the x-axis. So at most one real root, so the discriminant $(b^2 - 4ac)$ must be non-positive

Therefore,

$$4|\langle u, v \rangle|^2 - 4\|u\|^2 \|v\|^2 \leq 0$$

Law of Cosines Proof

We can also prove this with law of cosines.

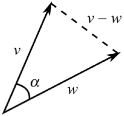

Figure: Wikipedia

Law of cosines then states

$$\|v\|^2 + \|w\|^2 - 2\langle v, w\rangle = \|v - w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos(\theta)$$

After some cancellation, we get

$$\langle v, w\rangle = \|v\|\|w\|\cos(\theta)$$

and since $\cos(\theta) \in [-1, 1]$, we get the desired bound

Triangle Inequality

We can now prove that the induced norm follows the triangle inequality:

Proof.

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle \\
&= \|u\|^2 + 2Re(\langle u,v\rangle) + \|v\|^2 \\
&\leq \|u\|^2 + 2|(\langle u,v\rangle)| + \|v\|^2 \\
&\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

□

Angles

Definition

For two vectors $u, v \in V$ with inner product $\langle \cdot, \cdot \rangle$, we can now define and angle between the two vectors

$$\theta_{u,v} = \arccos\left(\frac{\langle u, v \rangle}{\|u\|\|v\|}\right)$$

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Remark

If $\|x\| = 1$ *is well, we saw* x *is normal, and so if both hold then* x *is* **orthonormal**

Orthogonal implies Linear Independence

Proposition

Suppose $\{v_1, ..., v_m\}$ *are all pairwise orthogonal, with each* $v_i \neq 0$*, then that list is linearly independent*

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$$0 = \langle v_i, 0 \rangle = \langle v_i, a_1 v_1 + ... + a_m v_m \rangle = \overline{a_i} \|v_i\|^2$$

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- Since this holds for each a_i, we have linear independence

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Gram-Schmidt Orthogonalization

We now outline a procedure to convert any basis $\mathcal{B} = \{u_1, .., u_n\}$into an *orthonormal basis*:

1. Define $z_1 = \frac{u_1}{\|u_1\|}$ so that $\|z_1\| = \left\|\frac{u_1}{\|u_1\|}\right\| = \frac{\|u_1\|}{\|u_1\|} = 1$

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Remark

Note that the span of $\{u_1, ..., u_k\}$ is the same as the span of $\{z_1, ..., z_k\}$ because the are just linear combinations of each other. Furthermore, we still have linear independence from the previous proposition, so now we have an orthogonal basis, that is also normal.
This is extremely useful because if $v = a_1 z_1 + ... + a_n z_n$ then by Pythagorean Theorem/using orthogonality we get

$$\|v\|^2 = |a_1|^2 + ... + |a_n|^2$$